DSP

15/10/2015 الخميس د. محمد كرفسة محاضرة [3]

Properties of Convolution :

① Commutative Property of convolution

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-k)$$

$$= \sum_{K=-\infty}^{\infty} x(n-k)\, h(K)$$

Proof:-

$\boxed{\text{assume} \quad n-k=m}$

$K = n-m$

$K=-\infty \Rightarrow m=\infty$

$K=\infty \Rightarrow m=-\infty$

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-k)$$

$$= \sum_{m=-\infty}^{\infty} x(n-m)\, h(m)$$

$$y(n) = \sum_{K=-\infty}^{\infty} x(n-k)\, h(k)$$

هذه الخاصية تفيد في إثبات الخاصية الثانية

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② Causality of LTI :

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-k)$$

$$= \sum_{K=-\infty}^{-1} h(K)\, x(n-k) + \sum_{k=0}^{\infty} h(K)\, X(n-k)$$

$$= \left[\cancel{h(-1)}_{0}\, x(n+1) + \cancel{h(-2)}_{0}\, x(n+2) + \cancel{h(-3)}_{0}\, x(n+3) + \cdots\right]$$

$$+ \left[h(0)\, x(n) + h(1)\, X(n-1) + h(2)\, X(n-2) + \cdots\right]$$

System is causal when $h(n)=0,\ n<0$

$$\boxed{\text{The System is Causal if:} \quad h(n)=0\ ,\ n<0}$$

Ex: ① $h(n) = \{1, -2, \underset{\uparrow}{0}, 1/2, 3\}$ ⇒ non-Causal ($h(n) \neq 0,\ n<0$)

② $h(n) = \{\underset{\uparrow}{1}, -2, 0, 1/2, 3\}$ ⇒ Causal

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③ Stability of LTI:

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-k)$$

$$\text{if } \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Then the System is stable

Z-Transform:

- Cont. time system:-

time domain $\underset{LT^{-1}}{\overset{LT}{\rightleftarrows}}$ s-domain

- discrete time System

discrete time domain $\underset{Z.T^{-1}}{\overset{Z.T}{\rightleftarrows}}$ Z-domain

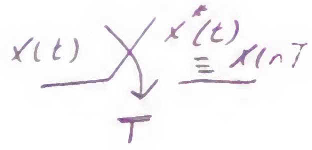
$z = e^{-Ts}$, T = Sampling Time

$$x(n) = \sum_{K=-\infty}^{\infty} x(K)\, \delta(n-k)$$

if $T \neq 1$

$$x(nT) = \sum_{K=-\infty}^{\infty} x(KT)\, \delta(n-k)T$$

⇓ L.T.



$$L[x^*(t)] = L[x(nT)]$$

$$= \sum_{K=-\infty}^{\infty} x(KT)\, e^{-KTS}$$

$$X^*(s) = \sum_{K=-\infty}^{\infty} x(KT)\, e^{-KTS} \Rightarrow (e^{+Ts})^{-K}$$

$$\text{Z.T} \downarrow \qquad z = e^{+TS}$$

$$\boxed{X(z) = \sum_{K=-\infty}^{\infty} x(KT)\, z^{-K}}$$

$$x(n) \xrightarrow{\text{Z.T.}} X(z)$$

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for a discrete time sequence :

$$x(nT) \equiv x(n) \Big|_{T=1\,sec}$$

the Z. transform for this sequence is

$$Z[x(n)] = X(z) = \sum_{K=-\infty}^{\infty} x(KT)\, z^{-K}$$

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad (T = 1\,sec)$$

Ex : find Z.T for $\delta(n)$

Solution

$$Z[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

for causal signals and system

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

we deal with LTI systems and causal system.

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \underbrace{x(0)}_{1} + \cancel{x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}} + \cdots$$

0

$$x(n) = \delta(n) = \begin{cases} 1 & , n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$Z[\delta(n)] = 1$$

Ex: find Z.T for $x(n) = u(n)$

Solution:-

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \cdots$$ متوالية هندسية منتهية الحد الثابت (z^{-1})

$$= \frac{1}{1 - z^{-1}}$$ $z^{-1} < 1$ ⇐ الشرط

$$X(z) = \frac{z}{z-1}$$ for $x(n) = u(n)$

* Region of convergence (ROC):

ROC: $z^{-1} < 1$ للمثال السابق:

$\Rightarrow z > 1$

Ex2: find Z.T for $x(n) = e^{-an}$

Solution:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-an} z^{-n} = 1 + e^{-a} z^{-1} + e^{-2a} e^{-2} + \cdots$$

$e^{-a} z^{-1} < 1$

$$= \frac{1}{1 - e^{-a} z^{-1}}$$

$$X(z) = \frac{z}{z-e^{-a}}$$

ROC: $e^{-a} z^{-1} < 1$ or $e^{a} z^{1} > 1$

Ex3: $x(n)=n$, find Z.T :-

Solution: $X(z) = \sum_{n=0}^{\infty} n z^{-n}$

$$X(z) = 0, \; z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots \quad \text{— } ①$$

$$z^{-1} X(z) = z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + \cdots \quad \text{— } ②$$

$$X(z) - z^{-1} X(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots$$

$$[1-z^{-1}] X(z) = \frac{z^{-1}}{1-z^{-1}}$$

ROC: $z^{-1} < 1$ or $z > 1$

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

Ex4: find Z.T for $x(n) = a^n$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$= \frac{1}{1-a z^{-1}}$$

$\boxed{\text{ROC: } a z^{-1} < 1}$

$$X(z) = \frac{z}{z-a}$$

Ex 5: $x(n)$ $\sin wn$, find $X(z)$

Solution: for $x(n) = \sin wn = \dfrac{e^{jwn} - e^{-jwn}}{2j}$

$\sin wt = \dfrac{e^{jwt} - e^{-jwt}}{2j}$

$\cos wt = \dfrac{e^{jwt} + e^{-jwt}}{2}$

$$X(z) = \sum_{n=0}^{\infty} \left[\frac{e^{jwn} - e^{-jwn}}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left[e^{jwn} - e^{-jwn} \right] z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} e^{jwn} z^{-n} - \sum_{n=0}^{\infty} e^{-jwn} z^{-n} \right]$$

$$= \frac{1}{2j} \Big[(1 + e^{jw} z^{-1} + e^{j2w} z^{-2} + e^{j3w} z^{-3} + \cdots)$$
$$- (e^{-jw} z^{-1} + e^{-j2w} z^{-2} + e^{-j3w} z^{-3} + \cdots) \Big]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{jw} z^{-1}} - \frac{1}{1 - e^{-jw} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{\cancel{1} - e^{-jw} z^{-1} \cancel{-1} + e^{jw} z^{-1}}{(1 - e^{jw} z^{-1})(1 - e^{-jw} z^{-1})} \right]$$

$$= \underbrace{\frac{1}{2j}}_{\sin} \left[\frac{z^{-1} (e^{jw} - e^{-jw})}{1 - \underbrace{(e^{jw} + e^{-jw})}_{\cos} z^{-1} + z^{-2}} \right]$$

$$X(z) = \frac{z^{-1} \left(\frac{e^{jw} - e^{-jw}}{2j} \right)}{1 - 2 \left(\frac{e^{jw} + e^{-jw}}{2} \right) + z^{-2}}$$

$$= \frac{z^{-1} \sin w}{1 - 2 \cos z^{-1} + z^{-2}}$$

$$X(z = \frac{z \sin w}{z^2 - 2z \cos w + 1}$$

Report for $\cos wn$ (Prove)

z - transform

$x(n)$	$\longrightarrow$ $X(z)$
$\delta(n)$	1
$u(n)$	$\frac{z}{z-1}$
$e^{\pm an}$	$\frac{z}{z-e^{\pm a}}$
a^n	$\frac{z}{z-a}$
n	$\frac{z}{(z-1)^2}$
$\sin \omega n$	$\frac{z \sin\omega}{z^2-2z\cos\omega+1}$
$\cos \omega n$	$\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$

Properties of z-transform:

① $z\left[\underset{\text{const.}}{a}\, x(n)\right] = a\, X(z)$

Ex: find z.T for $[3u(n)]$

$$= z.T\,[3] = 3\left(\frac{z}{z-1}\right)$$

② $z\left[x_1(n) \pm x_2(n)\right] = X_1(z) \pm X_2(z)$

③ $z\left[e^{\pm an}\, x(n)\right] = X(z)\Big|_{z = z e^{\mp a}}$ عكس الاشارة

Ex: find z.T of $n e^{2n}$

$$z\left[n e^{2n}\right] = \frac{z}{(z-1)^2}\Bigg|_{z = ze^{-2}}$$

$$= \frac{ze^{-2}}{(ze^{-2}-1)^2}$$

④ $z\left[n\, x(n)\right] = -z\,\frac{dX(z)}{dz}$

Ex: find z.T n^2

$$z\left[n^2\right] = z\left[n.\overset{x(n)}{n}\right] = -z\,\frac{d\left(\frac{z}{(z-1)^2}\right)}{dz}$$

$$z\left[n^2\right] = -z\,\frac{d\left(\frac{z}{(z-1)^2}\right)}{dz}$$

$$= -z\left[\frac{(z-1)^{\cancel{2}\,1}(1) - z(2)(\cancel{z-1})}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1) - 2z}{(z-1)^3}\right]$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}$$

(5) $z\left[a^n\, x(n)\right] = X(z)\Big|_{z=\frac{z}{a}}$

ex: $z\left[n a^n\right]$

$$= \frac{z}{(z-1)^2}\Bigg|_{z=\frac{z}{a}} = \frac{z/a}{(z/a-1)^2}$$

$$= \frac{az}{(za)^2}$$

Ex: $x(n) = a^n e^{2n}$, find $X(z$

Sol 1:-

$e^{2n} \xrightarrow{z.T} \frac{z}{z-e^2}$

$a^n a^{2n} \xrightarrow{z.T} \frac{z}{z-e^2}\Bigg|_{z=\frac{z}{a}}$

$$= \frac{z/a}{z/a-e^2} = \frac{z}{z-ae^2}$$

Sol 2:-

$a^n \xrightarrow{z.T} \frac{z}{z-a}$

$e^{2n}\cdot a^n \xrightarrow{z.T} \frac{z}{z-a}\Bigg|_{z=ze^{-2}}$

$$= \frac{ze^{-2}}{ze^{-2}-a} = \frac{z}{z-ae^{+2}}$$

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(6) Convolution in z-domain:

$$x_1(n) * x_2(n) \xrightarrow{z.T.} X_1(z)\cdot X_2(z)$$

Ex: $x_1(n) = 3\delta(n) + 2\delta(n+1)$

$x_2(n) = 2\delta(n) - \delta(n-2)$

find $X(z) = z\left[x_1(n) * x_2(n)\right]$

Solution

$X(z) = Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z)$

$$\boxed{\begin{aligned} X_1(z) &= 3 + 2z^{-1} \\ X_2(z) &= 2 - z^{-2} \end{aligned}} \Rightarrow X(z) = X_1(z) \cdot X_2(z)$$

$$= (3 + 2z^{-1})(2 - z^{-2})$$

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(7) $Z[x(n-m)] = z^{-m} X(z)$ Z.I.C

Ex: $x(n) \{1, 0, 1, 0.5, 2\}$

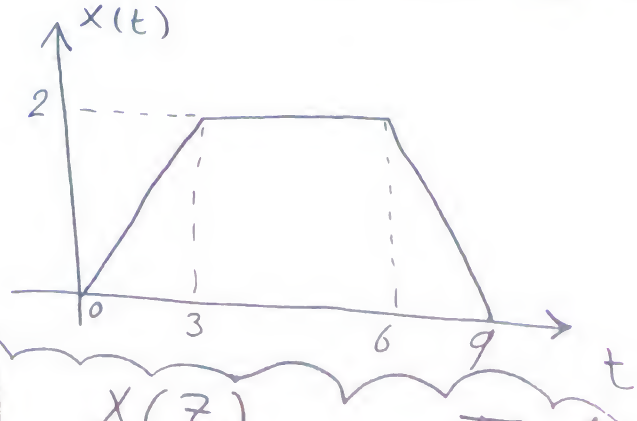
find $X(z)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \underbrace{x(0)}_{1} + \underbrace{x(1)}_{0} z^{-1} + \underbrace{x(2)}_{1} z^{-2} + \underbrace{x(3)}_{0.5} z^{-3} + \underbrace{x(4)}_{2} z^{-4}$$

$$= 1 + z^{-2} + 0.5 z^{-3} + 2z^{-4}$$

Report



find $X(z)$ $T=1$